# A Monotone Map on 2-Manifolds, Whose Image is Homeomorphic to Its Domain Space, A 2-Manifold (Combinatorial Topology)



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# A MONOTONE MAP ON 2-MANIFOLDS, WHOSE IMAGE IS HOMEOMORPHIC TO ITS DOMAIN SPACE, A 2-MANIFOLD.

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§0. The main results in this note are Theorem A (see §2) and Theorem B (see §3) which are extensions of Whyburn's theorem [1]; Any monotone mapping of a plane onto a plane is compact. In §1, extensions of a well-known Moore's theorem concerning a decomposition of a 2-sphere, are stated without proof, Theorem 1, 2, which are used to show Theorem A and B. Terminologies are explained also in §1, whose meanings will be found among Lemmata 1~10. Finally the author regrets that he has not enough informations concerning these problems: Anyone gets the same results?

§1. A map $f(X)=Y$ is <u>compact</u> <u>(connected)</u> iff the inverse image $f^{-1}(B)$ of any compact (connected) set B of Y is compact (connected).

Lemma 1. Suppose M is a connected compact n-manifold, $n>1$, and W is its connected open subspace with a totally disconnected complement $K=X-W$. Any compact and connected map $f(W)=W$, has a unique extension $g(M)=M$, which is also compact and connected, whose restriction on K is a homeomorphism on K.

A map $f(X)=Y$ is monotone iff the inverse image $f^{-1}(y)$ of any point $y$ of $Y$ is always compact and connected in $X$. A fundamental domain of a 2-manifold is a homeomorphic image of a connected open set in a plane.

Lemma 2. A subset $C$ in a 2-manifold $M$ is cellular iff $C$ is compact, connected and is contained in such a fundamental domain $W$ of $M$, that the complement $W-K$ is connected.

Lemma 3. Let $X$ be a locally compact Hausdorff space and $\{K_i \mid i=1, 2, \dots\}$ be its countable compact covering, then for any open set $U$ of $X$, there is an integer $n$ such that $U \cap \mathrm{Int}\, K_n \neq \emptyset$.

Lemma 4. A separable metric n-manifold $M$ has a countable connected open cover $M=\bigcup_{i=1}^{\infty} M_i$ such that the closure $\overline{M}_i$ is compact and $\overline{M}_i \subset M_{i+1}$ for each $i$.

Lemma 5. Given an injective map $f: M_1 \to M_2$ from a m-manifold without boundary $M_1$ into a m-manifold $M_2$, then the map $f$ is an imbedding and its image $f(M_1)$ is an open set in the interior of $M_2$.

A map $f(X)=Y$ is quasi-compact iff the set $B$ of $Y$, whose inverse image $f^{-1}(B)$ is closed in $X$, is also closed in $Y$. A map $f(X)=Y$ is upper semi-continuous (u.s.c.) iff for any open set $U$ of $X$, the set $\tilde{U}$ in X, defined by $\tilde{U}=\cup\{f^{-1}(y) \mid y\in Y,\ f^{-1}(y)\subset U\}$, is open in X.

Lemma 6. A map f(X)=Y is closed iff the map f is quassi-compact and u.s.c.

Lemma 7. Let f(X)=Y be a closed map. If X is a separable metric space, so is Y.

Lemma 8. Let the map f(X)=Y be quasi-compact such that the inverse image of any point of Y is connected. Then if X is locally connected, so is Y.

Lemma 9. If the map f(X)=Y is closed and monotone, then it is compact and connected.

Lemma 10. A monotone map f(X)=Y from a locally compact space X onto a Hausdorff space Y is U.s.c.

Lemma 11. Let f(X)=Y be a monotone map from a locally compact metric space X onto a Hausedorff space Y. For any compact set K in X, the inverse image $f^{-1}f(K)$ is also compact in X.

A disjoint closed cover G of a space X is called <u>a decomposition of a space X</u>, and its quotient space X'=X/G is called <u>a decomposition space</u> of X by G, where the projection $\phi$:X→X' is clearly quasi-compact. A decomposition G of a space X is <u>u.s.c.</u> iff the projection $\phi$:X→X/G is u.s.c; and it is <u>compact</u> <u>(connected)</u> iff each element of which is compact (connected); it is <u>non-separating</u> iff for any element $K \in G$, the complement $X-K$ is connected. A map $f(X)=Y$ induces naturally a decomposition of $X$, $G(f)=\{f^{-1}(y) \mid y \in Y\}$, andwe denote its decomposition space by $\phi: X \to X/G(f)$, where the combined map $h=f\phi^{-1}: X/G(f) \to Y$ is well defined, and bijective. It is clear that the map h is homeomorphism iff the map f is quasi-compact.

Moore's Theorem. Given a non-trivial decomposition $G(X)$ of a space $X$, where the space $X$ is a 2-sphere or 2-plane, which is monotone, u.s.c. and non-separating, then the decomposition space $X/G$ is homeomorphic to the space $X$.

Lemma 12. Given a 2-sphere $X$ and its connected open subset $W$ with a decomposition $G(W)$ which is u.s.c. and monotone. Let $K$ be the complement of $W$ in $X$, and $G(K)$ be its decomposition whose element is a component of $K$. Define the decomposition of a 2-sphere $X$, by $G(X)=G(W) \cup G(K)$, then the decomposition space $X/G(X)$ is a Hausdorff space.

Theorem 1. ([2]) Given a connected open subspace $W$ of a 2-sphere $S^2$ and its non-trivial decomposition $G(W)$, which is monotone, u.s.c. and non-separating. Then the decomposition space $W/G(W)$ is homeomorphic to $W$.

Theorem 2. Let M be a separable metric 2-manifold without boundary, G be its u.s.c. cellular decomposition. Then the decomposition space M/G is homeomorphic to M.

Suppose A is a subset of a space X. A point a of A is a cut point of A, iff the complement A- a is disconnected. A subspace A is a true cyclic element of X, iff it is maximal in a sense that it has no cut points. A cactoid is a locally connected continuum every true cyclic element of which is a 2-sphere.

Theorem 3. (Moore) Every monotone image of a 2-sphere, which is a locally connected continuum, is a cactoid and every cactoid is the image under some monotone mapping.

§2. Theorem A. Let W be a connected open subspace of a 2-sphere $S^2$, and M be a topological 2-manifold in the most large sense. Given a monotone map $f:W\to M$, whose image f(W) has a non-vacuous interior in M, then the image f(W) is in the interior of M, and homeomorphic to W, moreover, the map $f:W\to f(W)$ is closed, compact, and connected, and the natural decomposition space by f, g(W)=W' is homeomorphic to W.

Proof. The map $f:W\to f(W)$, is monotone, u.s.c. by Lemma 10, and non-separating (see the later argument), so the natural decomposition space by f, $g(W)=W'$ is homeomorphic to W by Theorem 1. Now the injective map $fg^{-1}:W'\to M$, is an imbedding and $fg^{-1}(W')=f(W)$ is open subspace of M, that is, f(W) is homeomorphic to W. (see Lemma 5.) Since the map $fg^{-1}:W'\to f(W)$ is a homeomorphism, the map $f:W\to f(W)$ is quasi-compact, so f is closed by Lemma 6. A closed monotone map $f(X)=Y$ is compact and connected by Lemma 9. Now we show that f is non-separating. Take a point $y\in f(W)$, then the inverse image $f^{-1}(y)=N$ is compact and connected in W, since the map f is monotone. The 2-manifold W has a connected open cover $W=\bigcup_{i=1}^{\infty}W_i$ such that $\overline{W}$ is compact and $\overline{W}_i\subset W_{i+1}$ for each $i$. (see Lemma 4) There is an integer $n_0$ such that $W_n\supset N$, $n>n_0$, because N is compact. Take an open set $U\subset \mathrm{Int}\ f(W)$, such that $\overline{U}$ is compact in Int f(W). Since $\overline{U}=\bigcup_{i=1}^{\infty} f(\overline{W}_i)\cap\overline{U}$ is a compact covering of a compact Hausdorff space, (see Lemma 3), there is an integer $n_1$, such that for any $n>n_1$, $f(\overline{W}_n)$ contains an open 2-cell $C^2$ of M. Choose an integer n, such that $n>n_0+n_1$. Since $H=f^{-1}f(\overline{W}_n)$ is compact by Lemma 11, the restriction $f:H\to f(\overline{W}_n)$ is a closed map, so H is connected by Lemma 9. Thus H is a continuum in W. There are a finite number of 2-disks in the 2-sphere $S^2$, say $D_1, D_2, \ldots, D_d$, whose union is denoted by $D=\bigcup_{i=1}^{d}D_i$, which satisfy that $D\cap H=\phi$, $(\bigcup_{i=1}^{d}\mathring{D}_i)\supset(S^2-W)$. We may assume that the boundary

$\partial D$ consists of a finite number of disjoint 1-spheres, say $S_1$, $S_2$, ..., $S_s$, that is $\partial D=\bigcup_{j=1}^{s} S_j$. For each j, the inverse set $f^{-1}f(S_j)\subset W$, is a continuum by the same argument for H, so the union $f^{-1}f\partial D=\bigcup_{j=1}^{s} f^{-1}fS_j$ is compact in W and has less than (s+1) components, Let Q be the component of the complement $W-(f^{-1}f\partial D)$, which contains a connected set H, then it is clear that $\overline{Q}$ is compact in W and $\partial Q$ has a finite number of components, which implies that the complement $(S^2-Q)$ has a finite number of components, say $E_1$, $E_2$, ... , $E_m$, $(m\leqq s)$, that is $S^2-Q=\bigcup_{i=1}^{m} E_i$. Since f has a connected decomposition, we know that $f^{-1}fQ=Q$, so the restriction $f:Q\to f(Q)$, is a quasi-compact map, because $f:\overline{Q}\to f(\overline{Q})$ is a closed map. (See Lemma 9.) Consider the decomposition space of the 2-sphere $S^2$, $\phi:S\to K$, by its monotone decomposition defined by $G(S^2)=\{E_1, \dots , E_m\}\cup\{f^{-1}f(x)\,|\,x\in Q\}$, which is a Hausdorff space by Lemma 12. So the map $\phi(S^2)=K$ is closed by Lemma 10 and 9, which implies K is a locally connected separable metric space which is also compact and connected. (See Lemmata 7 and 8.) After all the monotone image K of a 2-sphere is a cactoid, by Theorem 3. The map $h=f\phi^{-1}:\phi(Q)\to f(Q)$ is a homeomorphism because $f:Q\to f(Q)$ is quasi-compact, so the inverse image $h^{-1}(C^2)$ of an open 2-cell $C^2\subset f(Q)$, is a non-degenerate connected subset of K, which has no cut point of

$h^{-1}(C^2)$, whence we may say that the cactoid K has at least one $E_0$-set, namely one 2-sphere $\Omega \subset K$. There is a point $p \in \Omega$, such that $K-\{p\}$ is connected, because generally any simple link or $E_0$-set in a connected set X contains at most a countable number of cut points of X. Now we define a connected subset $K_0$ in K, by $K_0 = \phi(Q)-\{p\}$,

$K_0 = \phi(Q)-\{p\} = K-\{\phi(E_1), \ldots, \phi(E_m), p\}$, in which the subset $Z = \Omega-\{\phi(E_1), \ldots, \phi(E_m), p\}$ is closed and open, so we know that $K_0 = Z$, that is, K is a 2-sphere $\Omega$. The reason why the set Z is closed and open in $K_0$: It is clear that the closure of Z in K is contained in $\Omega$, because the compact set $\Omega$ is closed in a Hausdorff space K and $Z \subset \Omega$, which means that $\bar{Z} \cap K_0 = Z$, that is Z is closed in $K_0$. Next, the map from a 2-manifold Z without a boundary into a 2-manifold M, $h: Z \to M$, is injective, the image h(Z) is open in M, by Lemma 5, so h(Z) is also open in $h(K_0)$. So $Z = h^{-1}(hz)$ is open in $K_0$, because $h|K_0$ is a homeomorphism. Finally the closed monotone map $\phi: S^2 \to K$ is connected by Lemma 9, so the inverse image of a connected set $K-\phi f^{-1}(y)$, where K is a 2-sphere and $\phi f^{-1}(y)$ is a point of $\phi(Q)$, is connected, that is, the complement $S^2-f^{-1}(y)$ is connected, so $W-f^{-1}(y)$ is connected.

§3. Theorem B. Let $f: M_1 \to M_2$ be a monotone map, Int $f(M_1) \neq \phi$, from a separable metric 2-manifold $M_1$ without a boundary into a 2-manifold

$M_2$. If any element K of the monotone decomposition G of X, defined by $G=\{f^{-1}(y) \mid y \in f(M_1)\}$, is contained in a fundamental domain W of $M_1$, Then the map f is closed, $M \approx f(M_1)$ (homeomorphic) and $f(M_1) \subset \mathring{M}_2$.

Proof. There is such a compact set A in $M_1$ that Int $f(A) \neq \phi$ in $M_2$ and $f^{-1}f(A)=A$. Take a countable open cover $N=\{N_i\}$ of $M_1$ such that the closure $\overline{N}_i$ is compact and $\overline{N}_i \subset N_{i+1}$ for each i=1, 2, ..., and a 2-disk D in the interior of $f(M_1)$. Since $\{D \cap f(\overline{N}_i) \mid i=1, \ldots\}$ is a compact cover of a 2-disk D, there is an integer m such that the interior of $(D \cap f\overline{N}_m)$ in D is non-vacuous, that is Int $f(\overline{N}_m) \neq \phi$ in $M_2$, whence we define $A=f^{-1}f\overline{N}_m$, which is a compact set in $M_1$. There is a fundamental domain $W_0$ such that the image $fW_0$ is open in $\mathring{M}_2$, $f^{-1}fW_0=W_0$ and for any $K \in G_{W_0}=\{K \in G \mid K \subset W_0\}$, the complement $(W_0-K)$ is connected. For any element $K_\nu \in G$, we may choose such a fundamental domain $W_\nu$ containing $K_\nu$ as $f^{-1}fW_\nu=W_\nu$. Take a fundamental domain W of $K_\nu$, then $\widetilde{W}=\{K_\mu \in G \mid K_\mu \subset W\}$ is an open set of $M_1$, because G is u.s.c. decomposition of $M_1$. Let $W_\nu$ be the component of $\widetilde{W}$, which contains $K_\nu$. It is clear that $W_\nu$ be a desired one. There is an open set $V_\nu$ of $M_1$, such that $K_\nu \subset V_\nu \subset \overline{V}_\nu \subset W_\nu$, because a metric space $M_1$ is normal. Define $\widetilde{V}_\nu$ by $\widetilde{V}_\nu=\{K \in G \mid K \subset V_\nu\}$, then we have $A=\bigcup_\nu (A \cap \widetilde{V}_\nu)=\bigcup_{i=1}^{n} (A \cap \widetilde{V}_i)=\bigcup_{i=1}^{n} (A \cap cl\widetilde{V}_i) \subset \bigcup_{i=1}^{n} W_i$, because A is compact. There is an integer m, such that

Int $f(A)\cap$ Int $f(A\cap \mathrm{cl}\widetilde{V_m})\neq\phi$, in other words, Int $(fW_m)\neq\phi$ in $M_2$. Now $W_0=W_m$ is a desired one, by the Theorem A. For any $K_\nu\in G$, the complement $(W_\nu-K_\nu)$ is connected. Take a path $\sigma:[0,1]\to M_1$, such that $\sigma(0)\in W_0$ and $\sigma(1)\in K_\nu$. Since the set $f^{-1}f\sigma[0,1]$ is compact and connected, there is a sequence of fundamental domain $W_0, W_1, \ldots, W_n=W_\nu$ such that $f^{-1}f\sigma[0,1]\subset\bigcup_{i=0}^{m} W_i$ and $(\bigcup_{i=0}^{j} W_i)\cap W_{j+1}\neq\phi$ $(j=0,1,\ldots,n-1)$. It is clear that $W_0\cap W_1=f^{-1}f(W_0\cap W_1)$, so $G_{01}=\{K\in G\,|\,K\subset W_0\cap W_1\}$ is a decomposition of the intersection $W_0\cap W_1$, which is monotone, u.s.c. and non-separating. Using Theorem 1, it is known that $(W_0\cap W_1)\approx f(W_0\cap W_1)$ which is in $\mathring{M}_2$ and open in $M_2$. It implies that Int $fW_1\neq\phi$. By Theorem A we know that $f(W_1)$ is open in $\mathring{M}_2$ and for any $K\in G_1=\{K\in G\,|\,K\subset W_1\}$ $(W_1-K)$ is connected. After the same type of n arguments, we know that $(W_\nu-K_\nu)$ is connected. Finally by Theorem 2, the decomposition space $\phi:M_1\to M_1'=M_1/G$ is homeomorphic to $M_1$, and the map $h=f\phi^{-1}:M_1'\to M_2$ is a homeomorphism, which implies that $M_1\approx f(M_1)$, $f(M_1)\subset\mathring{M}_2$ and the map $f:M_1\to M_2$ is quasi-compact. Since the map $f:M_1\to M_2$ is also u.s.c., the monotone map is closed, that is compact and connected.

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